Tolerants

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Introduction

"What are the roots of a polynomial?" is one of the most natural and foundational questions in algebra. Far from being a purely algebraic concern, understanding polynomials and their roots plays a key role in areas such as topology and algebraic geometry.

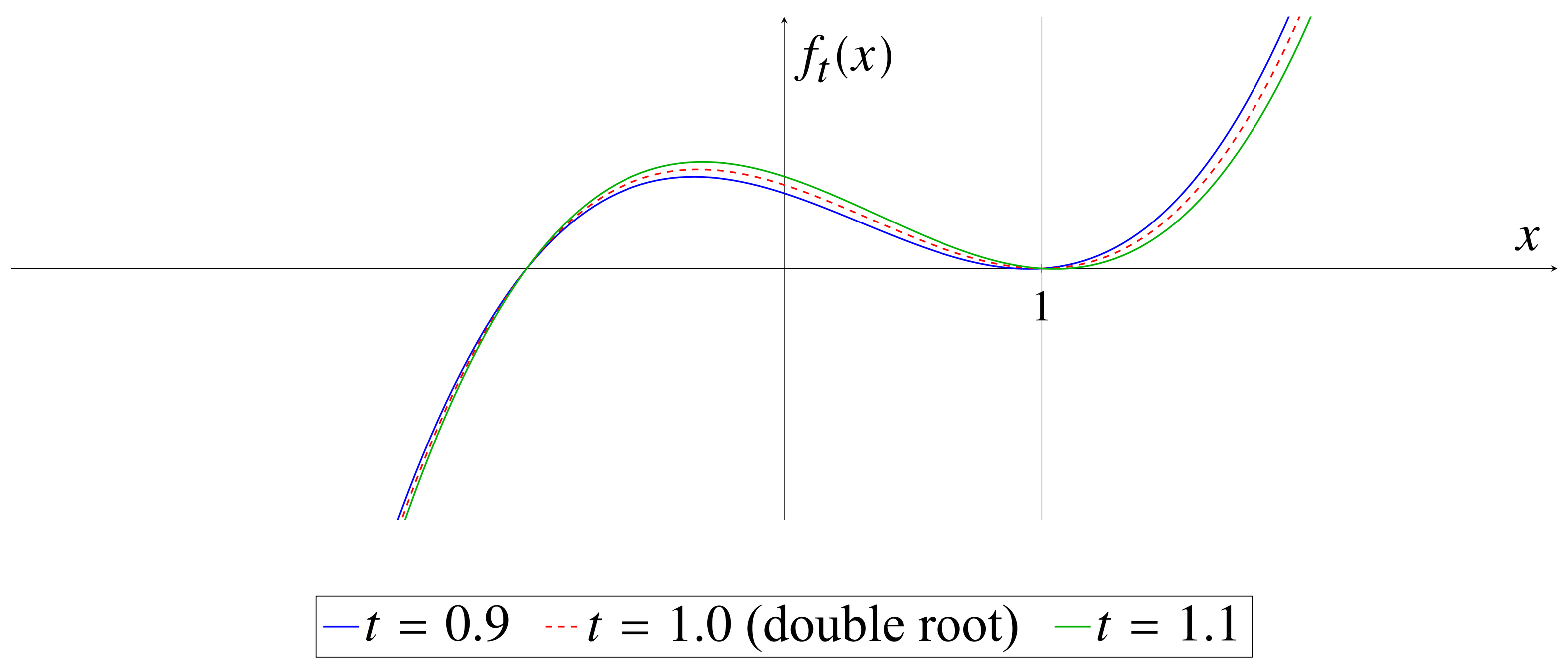

Figure 1. Polynomials $f_t(x) = (x-1)(x-t)(x+1)$ showing roots approaching collision at $x = 1$ when $t = 1$. The discriminant vanishes at $t = 1$, but the tolerant remains nonzero.

The discriminant is the usual tool that tells us how roots behave and when exactly they collide. But when roots repeat, the discriminant is always zero and tells us nothing else about the roots. This leads us to a new tool: a variant of the discriminant called the tolerant which does not vanish even when roots collide.

Definitions

What's the Discriminant?
Let $f(x) = a_n(x-r_1)^{m_1}\cdots(x-r_s)^{m_s}$. The *discriminant* is:

$$\mathrm{Disc}(f) = a_n^{2n-2}\prod_{i<j}(r_i - r_j)^2$$

- Detects if any roots collide.
- But fails (equals 0) when roots repeat!

What's the Duplicant?
We define the *duplicant* of the same polynomial as:

$$\mathrm{Dupl}(f) = a_n^{2n}\prod_{i<j}(r_i - r_j)^{2m_im_j}$$

- Also detects if roots collide.
- Works even with repeated roots (does not vanish).
- Equals a_n^2 times the discriminant when f is separable.

Meet The Tolerant

The *tolerant* of a polynomial can be defined as:

$$\mathrm{tol}(f) := \frac{\mathrm{dupl}(f)}{a_n^2}$$

Unlike the discriminant, the tolerant remains nonzero even when roots collide. And unlike the duplicant, it behaves more predictably under algebraic operations. For monic polynomials (such as those arising in our configuration space operation Σ_D), we have:

$$\mathrm{tol}(f) = \mathrm{dupl}(f).$$

Main Results

We prove two key theorems about the tolerant:

Theorem 1: Rationality
If $f(x) \in k[x]$, then

$$\mathrm{tol}(f) \in k^\times.$$

That is, the tolerant always stays within the field.

Theorem 2: Inversion Invariance
Let $f(x) \in k[x]$ with $f(0) \neq 0$, and define its reciprocal polynomial as

$$f^*(x) = x^{\deg f} f(1/x).$$

Then the set

$$T = \{f \in k[x] \setminus (x) : \mathrm{tol}(f) = \mathrm{tol}(f^*)\}$$

is a proper multiplicative subset of $k[x] \setminus (x)$.

This is particularly interesting because the discriminant is always invariant under polynomial inversion, but the tolerant isn't. It is perhaps more interesting that the set of polynomials for which the tolerant is inversion invariant is multiplicative.

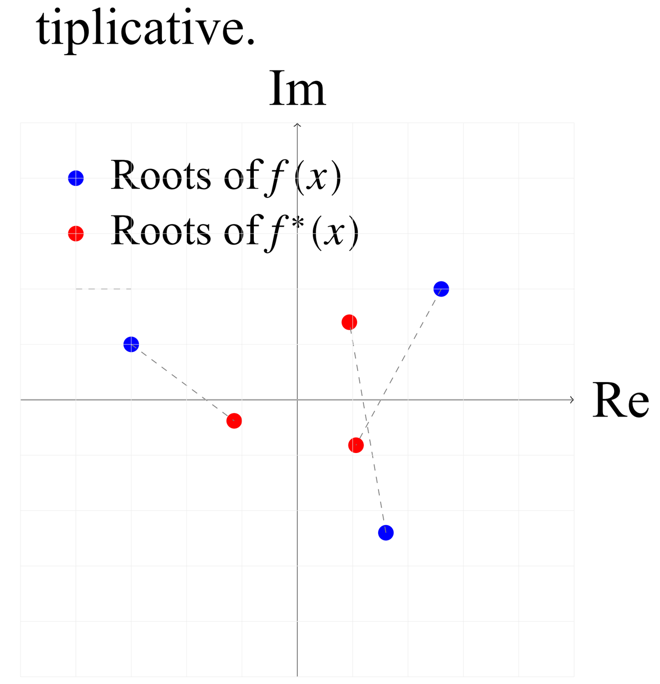

Figure 2. Reciprocal roots

Another key property of the discriminant is its invariance under translation and scaling of the polynomial. The tolerant also follows both properties which we show by some straightforward manipulation of variables. To be more specific, for any $\alpha \in k$, we have:

$$\mathrm{tol}(f(x+\alpha)) = \mathrm{tol}(f(x)) \text{ and}$$

$$\mathrm{tol}(f(\alpha x)) = \alpha^{n^2-2n+\sum_i m_i^2}\,\mathrm{tol}(f(x)).$$

Intuition Behind Proofs

Rationality. Why is the tolerant always in the field?

- Like the discriminant, the tolerant is ultimately built from coefficients.
- The discriminant's rationality comes from being the determinant of the Sylvester matrix (the resultant).
- Initially, we proved rationality for the tolerant by breaking the polynomial into irreducible factors:
  - For separable factors, the tolerant simplifies into products of discriminants.
  - For inseparable factors (in positive characteristic), we tweak using separable approximations and characteristic exponents.
- Later, we discovered a direct formula: a *resultant-based expression* for the tolerant, using only coefficients.
- So, we now have two proofs, one from factorization, and one from a matrix determinant, both confirming rationality.

Inversion Invariance. When does $\mathrm{tol}(f) = \mathrm{tol}(f^*)$?

- The tolerant almost behaves multiplicatively:

$$\mathrm{tol}(fg) = \mathrm{tol}(f)\cdot\mathrm{tol}(g)\cdot(\text{correction term}).$$

- Using this, we define: $S := \{f \in k[x] : \mathrm{tol}(f) = \mathrm{tol}(f^*)\}$.
- Then we prove:
  - S is closed under multiplication.
  - It includes palindromic polynomials and other natural families.
  - If $f \notin S$, then $\mathrm{tol}(f) \neq \mathrm{tol}(f^*)$, so S is a proper subset.
- So, inversion invariance doesn't hold everywhere, but it does on a special multiplicative submonoid.

Applications and Future Work

The duplicant appeared as part of a formula for unstable local degrees in motivic homotopy theory. It helped in tracking multiplicities of fixed points in $\mathbb{P}^1$-loop spaces with more precision. Since the tolerant is rational, it allows us to extend these local degree computations to points with complicated, non-rational coordinates. The tolerant refines how we track root collisions, making it easier to study fixed points and local behavior in unstable motivic homotopy theory.

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