Tolerants

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Roots of a Polynomial

What are the roots of a polynomial?
Degree 2:

$$x = \frac{-b \pm \sqrt{\mathbf{b^2 - 4ac}}}{2a}$$

$b^2 - 4ac$ tells us a lot!

- Real? Imaginary?
- Double root?

Roots of a Polynomial

Degree 3:

$$x = \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$
$$+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$
$$- \frac{b}{3a}.$$

Gets REALLY ugly...

Discriminant

Degree 3 Discriminant:

$$\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2.$$

Degree 4 Discriminant:

$$\begin{aligned}\Delta = 256a^3e^3 &- 192a^2bde^2 - 128a^2c^2e^2 + 144a^2cd^2e - 27a^2d^4 \\ &+ 144ab^2ce^2 - 6ab^2d^2e - 80abc^2de + 18abcd^3 + 16ac^4e \\ &- 4ac^3d^2 - 27b^4e^2 + 18b^3cde - 4b^3d^3 - 4b^2c^3e + b^2c^2d^2.\end{aligned}$$

Degree 5

Uh oh, no general formula for the roots...

However, 5th degree discriminant does exist.

Alternative Definition of Discriminant

Discriminant as a product of roots:

$$\Delta = a_n^{2(n-1)} \prod_{i<j} (s_i - s_j)^2.$$

- Well-known formulation!
- Removes dependence on general solution.

Tolerant

$$x^3 - 3x + 2 = (x-1)^2(x+2)$$

Distinct Roots: 1 (with multiplicity 2) and -2.

$$\Delta_0 := (r_1 - r_2)^{2m_1m_2}$$

$$\Delta_0 = (1-(-2))^{2\cdot 2\cdot 1} = 81$$

- Does not vanish when roots repeat. NEVER 0.

Formally Defining the Tolerant

Definition (Igieobo, McKean, Sanchez, Taylor, Wickelgren, '24)

Let $f(x) = a_n x^n + \ldots + a_0 \in K[x]$. Let $r_1, \ldots, r_s \in \overline{K}$ be the distinct roots of f, with multiplicities $m_1, \ldots, m_s$. The *tolerant* of f is the value

$$\mathrm{tol}(f) := a_n^{2n} \prod_{i<j} (r_i - r_j)^{2m_i m_j}.$$

Properties

Translation invariance

Properties

Translation invariance ✓

Properties

Translation invariance $\checkmark$

Homothety invariance

Properties

Translation invariance ✓

Homothety invariance ✓

Properties

Translation invariance ✓

Homothety invariance ✓

Rationality

Properties

Translation invariance ✓

Homothety invariance ✓

Rationality ✓

Properties

Translation invariance ✓

Homothety invariance ✓

Rationality ✓

Inversion invariance

Properties

Translation invariance $\checkmark$

Homothety invariance $\checkmark$

Rationality $\checkmark$

Inversion invariance $\times$

Properties: Translation Invariance

Proposition (A., H., M. '25)

$$\mathrm{tol}(f(x+\alpha)) = \mathrm{tol}(f(x))$$

$$f(x) = x^3 - 3x + 2 = (x-1)^2(x+2)$$
$$f(x+2) = x^3 + 6x^2 + 9x + 4 = (x+1)^2(x+4)$$

$$\mathrm{tol}(f) = 81$$
$$\mathrm{tol}(f(x+2)) = (-1-(-4))^{2\cdot 2\cdot 1} = 81$$

Properties: Homothety Invariance

Proposition (A., H., M. '25)

$$\operatorname{tol}(f(\alpha x)) = \alpha^{2n} \operatorname{tol}(f(x))$$

$$g(x) = x^3 - 3x + 2 = (x-1)^2(x+2)$$

$$\operatorname{tol}(g) = 81$$

$$g(2x) = 8x^3 - 6x + 2 = 2(x+1)(2x-1)^2$$

$$\operatorname{tol}(g(2x)) = 8^{2\cdot 3}(-1-(1/2))^{2\cdot 2\cdot 1} = 2^{2\cdot 3}\cdot 81$$

Properties: Inversion Invariance

$$f(x) = x^3 - 3x + 2 = (x-1)^2(x+2)$$

$$f^r(x) = x^3 f\left(\frac{1}{x}\right) = 2x^3 - 3x + 1$$

Is $\operatorname{tol}(f(x)) = \operatorname{tol}(f^r(x))$? Not always! Quite rarely, in fact.

$$\operatorname{tol}(f(x)) = 81$$

$$\operatorname{tol}(f^r(x)) = 108$$

Properties: Inversion Invariance Continued

When is it equal then?

$$h(x) = x^4 - 2x^2 + 1 = (x-1)(x+1)^2$$

Roots: 1, $m_1 = 2$ and -1, $m_{-1} = 2$.

$$h^r(x) = x^4 - 2x^2 + 1$$

$\mathrm{tol}(h(x)) = \mathrm{tol}(h^r(x))$ trivially!

Properties: Inversion Invariance Continued

Some more..

$$j(x) = x^5 + x^4 - 2x^3 - 2x^2 + x + 1 = (x-1)^2(x+1)^3 = j^r(x)$$

$$k(x) = x^4 - 2x^3 + 2x^2 - 2x + 1 = (x-1)^2(x-i)(x+i) = k^r(x)$$

Observations:

- $f(x) = f^r(x)$
- $a_n = a_0$ and the product of roots $=$ 1 or -1

Rationality

Rational meaning "in the field of coefficients"
The discriminant is rational:

- symmetric polynomial of roots
- Vieta's Formulas

Is the Tolerant rational?

$$f(x) = x^3 + x^2 - 8x + 12, \qquad \mathrm{tol}(f) = 256 \in \mathbb{Q}$$
$$g(x) = x^3 - 3x + 2, \qquad \mathrm{tol}(g) = 81 \in \mathbb{Q}$$

Alternative Formulations of the Tolerant

Factor f into irreducible polynomials

- The tolerant can be written in terms of discriminants.
- The discriminant is rational.

It follows that the tolerant is rational.

Theorem on Rationality

Theorem A., H., M. '25

The tolerant of any separable polynomial is rational.

$$\mathrm{tol}(f) = \alpha^{2n} \prod_{i=1}^{n} \Delta(f_i)^{m_i^2} \prod_{i<j} \left(\frac{\Delta(f_i f_j)}{\Delta(f_i)\Delta(f_j)} \right)^{2m_i m_j}$$

Thank you!

https://arxiv.org/abs/2506.22897